ENGINEERING CASE LIBRARY

ECL 1001

THE STORY OF EIGHTY-FOUR MINUTES

by

J.P. Den Hartog

Professor of Mechanical Engineering
Massachusetts Institute of Technology

Abstract

Maximilian Schuler, later a professor of dynamics at Göttingen, developed the gyroscopic ship's compass in the early 1920's and was the first to show that the error due to sideways acceleration of the ship would disappear if the compass could be tuned to a period of 84 minutes.

This case presents the development of this concept starting from the single particle pendulum, the physical pendulum, and the gyro pendulum. In addition, it is shown that a number of other phenomena are subject to the same 84 minute period. A stone falling along a diameter through the center of the earth takes 84 minutes to return to its original position, assuming uniform density of the earth. The same period occurs for a body moving along a frictionless "horizontal" track, also for a body moving without friction in a straight tunnel connecting any two points on the surface of the earth, and finally for a satellite going around the earth in a low orbit. It is shown that the same period occurs for a "roof top" satellite around any other celestial body having the same density as the earth.

The most important engineering application of the 84 minute pendulum is the inertial guidance system for spacecraft, missiles, airplanes, and ships. The case illustrates the effect of initial guidance platform misalignment upon the guidance error of an inertial system.

Prepared with the support of the National Science Foundation.

I. Introduction

Around the year 1920 the gyroscopic ship's compass was developed. This is a marvelous device which senses two things: the direction of the gravitational vertical and the once per 24 hour rotation of the earth about its axis. From these two inputs the compass automatically points horizontal and North and it has the astounding property of being self correcting: if the compass is disturbed and started in a position considerably different from the horizontal and North it will slowly drift back to the desired correct position. When the first compasses were built and placed in service it was found that they became completely useless on a rolling ship proceeding in a NE - SW direction. This effect was called the "intercardinal rolling error"; 'cardinal' meaning the two principal compass directions NS or EW and 'intercardinal' being between those. After exhaustive research the effect was found to be due to the sidewise accelerations of the compass caused by the rolling motion of the ship.

The phenomenon was very complicated and a good practical solution was found in the 1920's by Max Schuler, the developer of the device, and later professor of dynamics at the famous University of Göttingen in Germany. On (complicated) theoretical grounds he deduced that the error would disappear if the compass was tuned to a natural period of 84 minutes. Subsequent compasses built to this specification operated satisfactorily on rolling ships on an "intercardinal" course. The term "Schuler tuning" for the compass came into general use then. Since the physical and mathematical explanation of the effect as well as the constructional details of the gyrocompass are very complicated, a simpler "explanation" was devised. The 84 minutes arise from the formula

$$\omega = \sqrt{g/R}$$

where $g = 32.2\ \text{ft/sec}^2$ and R = the radius of the earth, 4000 miles. The formula for the natural frequency of a simple pendulum is

$$\omega = \sqrt{g/\ell}$$

so that 84 minutes is the "period of a pendulum with a length equal to the radius of the earth". A short pendulum with ℓ equal to a few feet, of which the upper point of support is accelerated horizontally will swing back and forth and hence its string will not be vertical. This, when applied to a compass will give a wrong indication of verticality to the device from which the operational error follows. Now if the pendulum is 4000 miles long, consisting of a heavy point mass at the center of the earth and the upper end of the (massless) string at the surface (the earth not interfering with the string) it is "clear" that we can move the top of the string horizontally in any way we like, and the string being massless will always remain radial, i.e., exactly vertical. This "simple explanation" is misleading or no explanation at all for all sorts of reasons. Aside from the fact that the earth would interfere with the string or that the whole thing is physically unrealizable, the value of

g at the center of the earth is zero, so that the string would be without tension. The formula $\omega = \sqrt{g/\ell}$ is true only for pendulums in which the length of the string is very small with respect to R, etc., etc.

However, it is physically possible (although not practically feasible) to construct a mechanical pendulum of small size (3 feet or so) which has a period of 84 minutes. Such a pendulum has indeed the property that when its point of support is accelerated horizontally, it will not swing, but always remain exactly vertical. This property is true completely irrespective of the constructional details of the pendulum, whether it contains a gyroscope or not. The ship's gyrocompass is a very complicated pendulum indeed, but it still obeys this general property. It is no exaggeration to state that the "Schuler tuning" is essential to the proper operation of the compass, and indeed made the ship's compass practically possible.

This was the 1920's. Since that day the 84 minute formula has cropped up repeatedly in a variety of problems, some academic without practical significance and others with spectacular application. These appearances of the 84 minute value ($T = 2\pi\sqrt{R/g}$) are the subject matter of this little essay. We will discuss the following topics:

a) Non sensitivity of an 84 minute "physical pendulum", (i.e., a rigid body rotatable in a vertical plane about one ot its points) to horizontal acceleration of the point of support.

b) The same for a gyroscopic pendulum.

c) If a hole be drilled clear through the (ideal, uniform) earth from the North Pole to the South Pole with the center of the earth in the middle, and a stone be dropped into it at the North Pole, it will fall, reach the South Pole after 42 minutes and be back home after 84 minutes.

d) If a "horizontal" straight track be built on the surface of the earth, which is truly horizontal only at its center point and on account of the earth's curvature is not horizontal away from the center, and if a particle can slide without friction on that track, it will oscillate about the track center with a period of 84 minutes (for small amplitudes $x_o \ll R$).

e) If a straight tunnel be built through the ideal, uniform earth, not along a diameter but at an arbitrary angle α with respect to a diameter, then a particle released at the earth's surface to slide without friction on a straight track in the tunnel will reappear at the point of origin in 84 minutes. The property c) is a special case of this one, for $\alpha = 0$.

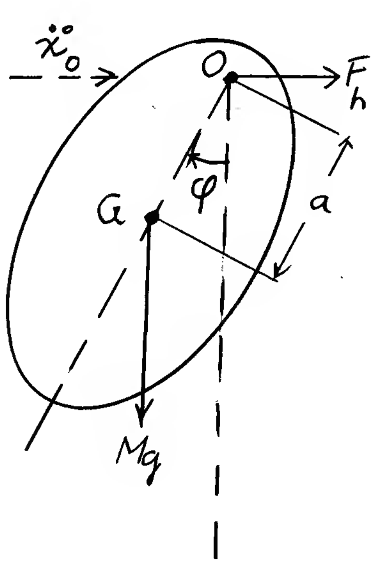
f) In our space age of satellites, the period of a "roof top" satellite, i.e., a satellite with a circular orbit at height h above the earth, when $h \ll R$ (like nearly all satellites up to now) has a period of 84 minutes.

g) A small man-made satellite in a circular, roof-top, orbit about the moon or any other planet or star will have an 84 minute period if the density of that star is the same as that of the earth. If not, the 84 minutes have to be divided by the square root of the density ratio. Since the moon's density is 0.607 of that of the earth, a satellite close to the moon will have a period of $84/\sqrt{.607} = 108$ minutes.

h) The properties c), d), and e) remain true for any other (ideal, uniform) planet if it has a density equal to that of the earth. Otherwise the same correction factor as g) has to be applied.

i) The gyroscopic ship's compass becomes useless if operated close to the earth's poles. In recent years, for submarine operation under the North polar ice, the compass has been superseded by "inertial navigation" or 'inertial guidance' apparatus. This does not contain a pendulum or any other gravity sensing device, and hence would be thought not to be listening to any formula $T = 2\pi\sqrt{R/g}$, containing the letter g. Nevertheless the appartus contains a table or platform with a pair of accelerometers (one EW, one NS) which must be mounted exactly horizontally at the start of the voyage and is kept horizontal afterwards by proper signals from a computer in the device. If an error is made in this initial adjustment, and the voyage is started with a small error angle between an accelerometer and the horizontal, this angle, by the nature of the servomechanism arrangement, will oscillate about zero (true horizontal) with a period of 84 minutes. This then is a case of automatic Schuler tuning, which is just as important to the practical operation of an inertial guidance system on the earth surface (for ships) as the man made Schuler tuning is for the ship's gyroscopic compass.

II. The Physical Pendulum



Let a pendulum be suspended at O and let that point O be given a horizontal acceleration $\ddot{x}_o$ to the right by a suitable horizontal force F_h. If the angle of swing φ is small enough and the point O moves horizontally only, the vertical acceleration of G is small of second order (i.e., proportional to φ^2), and hence the vertical component of the support force through O differs from Mg by a negligible amount only.

Write the two Newton equations of G:

$$\leftrightarrow \; F_h = M\ddot{x}_G = M(\ddot{x}_o - a\ddot{\varphi})$$

$$\circlearrowleft \; I_G\ddot{\varphi} = M\rho_G{}^2\ddot{\varphi} = F_h a - Mga\varphi$$

Eliminate the force F_h by substituting the first into the second equation:

$$M\rho_G{}^2\ddot{\varphi} + Mga\varphi = a\left[M(\ddot{x}_o - a\ddot{\varphi})\right]$$

$$\therefore \; \ddot{\varphi}(\rho_G{}^2 + a^2) + ag\varphi = a\ddot{x}_o \qquad \therefore \; \rho_o{}^2\ddot{\varphi} + ag\varphi = a\ddot{x}_o$$

$$\text{or} \quad \ddot{\varphi} + \frac{ag}{\rho_o{}^2}\varphi = \frac{a}{\rho_o{}^2}\ddot{x}_o$$

From the left hand side we see that the natural frequency of the physical pendulum is

$$\omega^2 = ag/\rho_o{}^2 \tag{1}$$

The φ-motion is forced by the right hand member. If the pendulum starts from rest ($\varphi = 0$) it is seen that the angle φ is subjected to an initial acceleration

$$\ddot{\varphi} = \left[\frac{a}{\rho_o{}^2}\right]\ddot{x}_o \tag{2}$$

proportional to the acceleration $\ddot{x}_o$. The pendulum thus will start to "hang back" and in course of time t the $\ddot{\varphi}$ will develop a

$$\varphi = \ddot{\varphi}t^2/2.$$

Now comes the trick. The point of suspension 0, experiencing $\ddot{x}_o$ will not stand still but in the course of time will develop a displacement x_o, which brings the pendulum to a different point on earth, where the local vertical is no longer parallel to the vertical at the starting point. Let the angle between the two verticals be ψ; then we have the geometric relation:

$$x_o = R\psi \tag{3}$$

where R is the radius of the earth. Satisfy yourself that the angle ψ is in the same direction as the "hang back" angle φ, and if by chance

$\varphi = \psi$ then the pendulum in its new location x_o is hanging exactly "vertical". Now differentiate (3) twice and substitute it into (2):

$$\ddot{\varphi} = \frac{a}{\rho_o^2} R\ddot{\psi}$$

In order for the pendulum to remain "vertical" in its new location φ must equal ψ, and since both start from zero $\ddot{\varphi}$ must equal $\ddot{\psi}$ or aR/ρ_o^2 must be unity.

$$\frac{aR}{\rho_o^2} = \frac{ag}{\rho_o^2}\frac{R}{g} = \omega^2 \frac{R}{g} = 1 \qquad (4)$$

Hence when the pendulum is tuned to $\omega^2 = g/R$ (which means that the dimensions be so chosen that $g/R = ag/\rho_o^2$, the hang back angle φ due to the acceleration will be equal to the angle ψ necessary to keep the pendulum "vertical" in its new location.

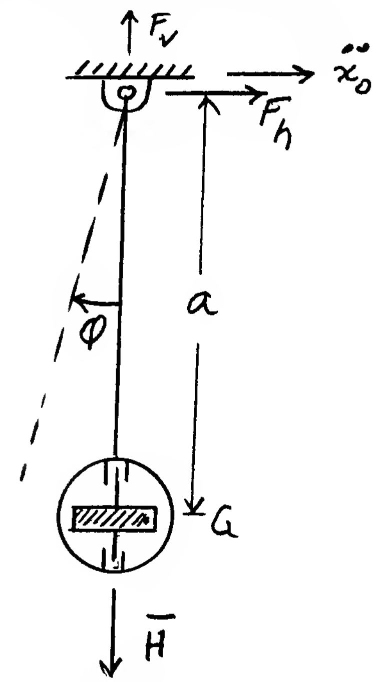
Problems

1. Satisfy yourself that $\omega^2 = g/R$ is equivalent to a period of 84 minutes.

2. Suppose the pendulum to be a uniform rod of 3 feet length. For proper Schuler tuning:
   a) What is the distance a between 0 and G?
   b) Suppose the hinge at 0 to be a small, well lubricated pin of very small radius $\rho_o/1000$. What friction coefficient in the pin will freeze this pendulum at angle φ_o?

3. In this argument the equations were written in an absolute or Newtonian coordinate system. Derive the result by writing the equations in a coordinate system in which point 0 remains fixed and hence is subjected to an acceleration $\ddot{x}_o$

III. A Gyroscopic Pendulum



Consider a pendulum supported at 0 by a frictionless ball joint, allowing angular motions in any direction, carrying a gyrowheel with its spin axis along the pendulum stem. Let the gyrowheel mass be M, its moment of inertia about the spin axis be $I = M\rho^2$, and let its angular momentum $H = I\Omega = M\rho^2\Omega$, where Ω is the constant angular speed, and the vector is directed downward. For simplicity assume that the mass of the pendulum stem and non-rotating gyroframe is negligible with respect to the rotating gyro mass M.

The point of support 0 of the gyropendulum is accelerated to the right by $\ddot{x}_o$, caused by a suitable force F_h. This will cause angular motions of the pendulum stem; not only a hang back angle φ as in the previous problem, but also an angle θ in a plane perpendicular to the paper (and to the $\ddot{x}_o$ direction). This angle θ we will call positive when the gyrowheel G deflects into the paper away from us. Again, for small angles φ and θ the vertical displacement and acceleration of G are second order small in φ or θ and hence negligible.

$$F_h = M\ddot{x}_G = M(\ddot{x}_o - a\ddot{\varphi}) \qquad (1)$$

$$F_v = Mg = \text{constant} \qquad (2)$$

When the pendulum finds itself in the general position φ, θ, the moments of the external forces about the center of gravity are:

in the φ - plane and $+\theta$ axial direction: $F_h a - Mga\varphi$

in the θ - plane and $+\varphi$ axial direction: $+Mga\theta$

The rate of change of angular momentum to which these moments must be equated are:

in the $+\varphi$ direction: $H\dot{\varphi}$

in the $+\theta$ direction: $H\dot{\theta}$

Hence, substituting equation (1), the equations of motion are

$$H\dot{\theta} = M(\ddot{x}_o - a\ddot{\varphi})a - Mga\varphi$$

$$H\dot{\varphi} = \mp Mga\theta$$

Before proceeding with these we remark that a gyroscope is very tenaceous in holding its angular position. Hence $\dot{\varphi}$ and $\ddot{\varphi}$ are very small, and we suspect that in the parenthesis of the first above equation $a\ddot{\varphi}$ is much smaller than $\ddot{x}_o$. We thus proceed to neglect $a\ddot{\varphi}$ with respect to $\ddot{x}_o$, subject to later verification. The equations then simplify to

$$H\dot{\theta} = M\ddot{x}_o a - Mga\varphi \tag{3}$$

$$H\dot{\varphi} = Mga\theta \tag{4}$$

Eliminate θ by substituting (4) into (3):

$$\ddot{\varphi} + \left(\frac{Mga}{H}\right)^2 \varphi = \left(\frac{Mga}{H}\right)^2 \frac{\ddot{x}_o}{g} \tag{5}$$

In the absence of acceleration $\ddot{x}_o$ we see that the natural frequency of the gyropendulum is:

$$\omega_n = \frac{Mga}{H} \tag{6}$$

Now, when starting from rest ($\varphi = 0$) the initial $\ddot{\varphi}$ acceleration is found from Equation (5), in which the second term is zero:

$$\ddot{\varphi} = \omega_n{}^2 \frac{\ddot{x}_o}{g}$$

This is the "hang back" angle. In order for the pendulum to remain vertical afterwards we found in the previous section that this angular acceleration has to be

$$\ddot{\varphi} = \frac{\ddot{x}_o}{R}$$

so that Schuler tuning requires that $\omega_n{}^2 = g/R$ as before, or:

$$\frac{Mga}{H} = \sqrt{g/R} \tag{7}$$

Now we like to know what this tuning formula means physically; in other words what is the required pendulum length a in terms of the other dimensions:

$$a = \frac{H}{Mg}\sqrt{\frac{g}{R}} = \frac{M\,\rho^2\Omega}{M\sqrt{gR}} = \rho\,\frac{\rho\Omega}{\sqrt{gR}}$$

In this $\rho\Omega$ is the peripheral speed of the gyrowheel at the radius of gyration which we make as high as possible, short of exploding the wheel by centrifugal force. A good figure for this speed is about 800 ft/sec. Hence

$$\frac{a}{\rho} = \frac{800}{\sqrt{gR}} = \frac{800}{\sqrt{32 \times 4000 \times 5280}} = 3\ \%$$

The pendulum length thus must be made very short: 3 % of ρ or about 1 % of the gyrowheel diameter. However this length is still a practical possibility, unlike the length found in Problem 2a) for the physical pendulum.

Now we are ready to verify if our previous assumption that $a\ddot{\varphi} \ll \ddot{x}_o$ is justified:

$$\frac{a\ddot{\varphi}}{\ddot{x}_o} = 0.03\rho\,\frac{\ddot{x}_o/R}{\ddot{x}_o} = 0.03\frac{\rho}{R}$$

Here ρ is a few inches and R is 4000 miles, so that the simplification is amply justified.

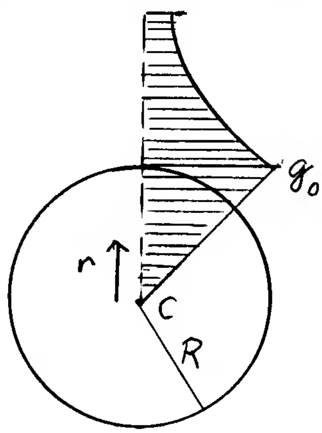
Problem 4

This is Problem 2a) for the gyropendulum. Find what friction coefficient in the Ball joint (with radius $\rho/1000$) will freeze the apparatus, i.e., prevent the first θ motion at the onset of an acceleration.

The actual gyroscopic ship's compass is a very complicated device. If its natural period is tuned to 84 minutes it becomes insensitive to accelerations of the base on which it is mounted. A complete explanation of this property takes some time and would disrupt the continuity of this essay. However it is the original structure for which the 84 minute period was devised and hence it is of importance. Therefore the explanation is given as an Appendix only (page 18).

IV. The Diametral Hole Through the Earth

Before we can work this problem we must know the value of g at various depths in the earth. It is obvious that g cannot be constant throughout, because for reasons of symmetry g must be zero at the center of the earth. Newton's gravitational law $F = \text{const.}\ m_1 m_2/r^2$ is stated in terms of two particles (i.e., point objects without size) attracting each other. For larger objects the force can be calculated by integration over all particles. The results of such an integration for the case of uniform density distribution (which is not simple: Problem 5 and 6) are shown in the figure, where the value of g is plotted as a function of distance from the center (of course the direction of g is always radial). Inside the earth the relation is linear or $g = g_o r/R$ where $g_o = 32.2$ ft/sec^2 and R = 4000 miles. The actual density distribution in the earth is far from uniform; the center core is more than twice as dense as the outer mantle, which makes the actual g distribution a non-linear function of r. Here we will discuss a uniform earth only.



Outside the earth the relation is as if the earth were replaced by a single concentrated point mass at its center:

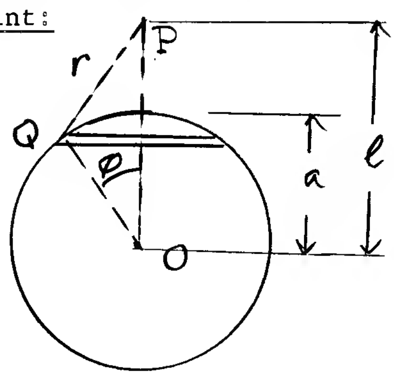
$$g = g_o\ (R/r)^2$$

This is true for any symmetrical density distribution f(r).

Problem 5

Prove that the Newtonian gravitational attractive force between a mass particle and a thin walled hollow sphere is zero when the particle is located inside the sphere, while for a particle outside the sphere the force is equal to that for the case that the entire sphere mass is concentrated in its center.

Hint:



Find the force between P and an element at Q in terms of the letters r and φ. Integrate over the thin ring $d\varphi$ for constant φ and constant r. This semi-integrated force by symmetry is directed along OP. Then write the integral over the sphere for φ from 0 to 180°. The evaluation of this integral is very messy, but it is much simplified if you first express r as a function of φ; then differentiate this relation and substitute the result

into the integral, which then becomes an integral of $f(r)\,dr$, not containing the letter φ.

Problem 6

With the theorem of Problem 5 derive the fact that for a solid sphere of constant density the gravitational acceleration g is a linear function of the radius, rising from zero at the center to g_o at the surface.

Now the problem of the falling stone. If the hole is along the polar axis of the earth, the rotation of the earth is irrelevant. At depth r from the center the force is $mg = mg_o r/R$ towards the center. Hence:

$$m\ddot{r} = -mg_o r/R$$

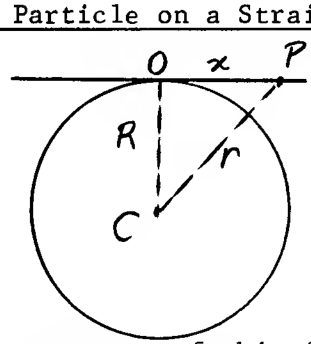
or

$$\ddot{r} + \frac{g_o}{R} r = 0$$

This is the simple vibration equation and the stone will execute harmonic vibrations with a frequency $\omega_n^2 = g_o/R$ about the center. The frequency is independent of the amplitude and $r_{max} = R$ is the special case where the stone takes 42 minutes to go from one surface pole to the other.

If the hole is not passing through the poles, Coriolis effects from the earth rotation come in and the free unrestrained path of motion will not be a straight line. If a straight, frictionless guide is provided the above answer is true for a diametral hole at any angle with the polar axis.

V. Particle on a Straight Surface Track



The gravity force on a particle m at P, directed along PC is:

$$mg = mg_o \frac{R^2}{r^2}$$

The component of this force along OP is x/r times as small

$$F_{hor} = mg_o \frac{R^2 x}{r^3} = mg_o \frac{R^2 x}{(R^2 + x^2)^{3/2}} = m\frac{g_o}{R} \frac{x}{(1 + x^2/R^2)^{3/2}}$$

Hence the equation of track motion is for no friction:

$$\ddot{x} + \frac{g_o}{R_o} \frac{x}{(1 + x^2/R^2)^{3/2}} = 0$$

a nonlinear equation giving a natural frequency of vibration depending on the amplitude x_{max}. For small vibrations $x \ll R$, this linearizes to:

$$\ddot{x} + (g_o/R)x = 0$$

so that for small amplitudes (i.e., x less than 10 miles) the natural frequency is $\omega^2 = g_o/R$ and the period is 84 minutes.

Problem 7

a) For a maximum amplitude of 1 mile each way, what is the maximum force component along the track on the particle?
b) For what friction coefficient will this motion be completely prevented at 1 mile amplitude?

Problem 8

Taking the complete non-linear equation of motion, by making the substitution:

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{d\dot{x}} \cdot \frac{dx}{dt} = \dot{x}\left(\frac{d\dot{x}}{dx}\right)$$

(which is usual with non-linear second order differential equations) derive the first integral:

$$\dot{x}^2 = 2g_oR\left[\frac{1}{\sqrt{1-(x/R)^2}} - \frac{1}{\sqrt{1-(x_o/R)^2}}\right]$$

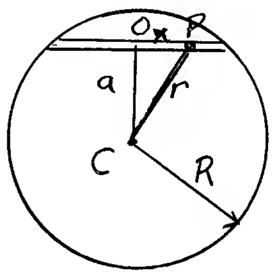
where x_o is the maximum amplitude of vibration. The "phase plane diagram" plots $\dot{x}/x_o\sqrt{g_o/R}$ vertically against (x/x_o) horizontally. Prove that for $x_o/R \ll 1$ this diagram is a unit circle. Then numerically plot this 'phase curve' for the highly non-linear case $x_o = 2R$, showing a distorted circle.

Problem 9

By the standard method of non-linear vibration carry out the second integration of the previous problem with the following result for the period of vibration T:

$$T = 2\sqrt{2}\sqrt{\frac{R}{g_o}}\sqrt[4]{1+(x_o/R)^2}\int_{x_o/R}^{0}\frac{\sqrt[4]{1-\xi^2}\cdot d\xi}{\sqrt{\sqrt{1+\left(\frac{x_o}{R}\right)^2}-\sqrt{1+\xi^2}}}$$

VI. Motion in Tunnel at Arbitrary Angle *



Let a tunnel be cut through the earth so that its center point O is at distance a from the earth's center C. Consider a particle m at P sliding without friction along the tunnel. Let OP = x and CP = r. The gravity force on m, directed along PC is $mg = mg_o r/R$. The component of this force along the track is x/r times as small or $mg_o x/R$. This force equals $-m\ddot{x}$, the minus sign is because the force acts in the negative x direction. Hence:

$$m\ddot{x} = -mg_o x/R$$

$$\ddot{x} + (g_o/R)x = 0$$

and the natural frequency of oscillation is $\omega = \sqrt{g_o/R}$ independent of the maximum amplitude of vibration.

VII. Earth Satellites

Let a satellite move in a circular orbit at height h above the earth's surface. It is in d'Alembert equilibrium when the centrifugal acceleration $\omega^2 r = \omega^2(R + h)$ equals the gravitational acceleration

$$g = g_o\left(\frac{R}{r}\right)^2 = g_o \frac{R^2}{(R + h)^2}$$

Equating these two quantities gives

$$\omega^2 = g_o \frac{R^2}{(R + h)^3}$$

which determines the period of the satellite. For a roof top satellite $h \ll R$ and the above expression reduces to $\omega^2 = g_o/R$ with an 84 minute period. The above expression can be written as:

$$\omega = \sqrt{\frac{g_o}{R}} \cdot \left(\frac{R}{R + h}\right)^{3/2}$$

---

* See *Time* magazine article, of February 11, 1966 reproduced on page 26.

The period of the satellite becomes larger than 84 minutes at increased heights h, and is found from 84 minutes by multiplying that quantity by

$$(R/R + h)^{3/2}.$$

For example if we desire a 24 hour satellite for communication purposes we have

$$\frac{24 \text{ hours}}{84 \text{ minutes}} = 17.2 = \left(\frac{R + h}{R}\right)^{3/2}$$

$$\frac{R + h}{R} = (17.2)^{2/3} = 6.6 \quad \text{and} \quad h = 5.6R = 22{,}400 \text{ miles}$$

Problem 10

What is the distance between the center of the earth and the center of the moon, expressed in earth radii?

VIII. Moon and Planet Satellites

All problems discussed so far have been with respect to the earth, assumed to be of uniform density. The answer to all has been so that we can conclude that the 84 minute period remains at that value for all other moons, planets or suns for which the ratio g_o/R is the same as on earth. Now let us look at a point in the interior of the earth and divide the earth into a solid sphere up to our point and a hollow sphere around it. Our point is a surface point for both, being on the inner surface of the hollow sphere. The g - value at our point is the integral of all attractions to all points. Now by the theorem of Problem 5 the g at our point caused by the outer hollow sphere is zero, and whatever g there is, is caused entirely by the solid inner core.

Since in the interior of the (uniform density) earth $g = g_o r/R$, this means that we can shave off the earth a crust of any thickness we like, and on the surface of the remaining core $g/r = g_o/R_o$.

Hence we conclude that the 84 minute period in all previous problems is preserved for all planets or moons which have the same density as the earth. For a planet or moon of less density, the value of R is the same, but g_o is diminished proportional to the density. Therefore the value of ω^2 is directly proportional to the density of the planet only, and the period T is inversely proportional to the square root of the density.

The moon's density is 0.607 times the density of the earth. Hence the characteristic period for the moon (for all applications discussed here) is

$$84 \text{ minutes } \frac{1}{\sqrt{.607}} = 108 \text{ minutes}$$

Problem 11

Assume that the earth is made of two materials, an inner spherical core and an outer hollow spherical mantle. Assume that the core density is twice the mantle density. The values of R and g_o are obviously the ones we observe. The outer radius of the core is 0.618 R.

a) Find how the diagram of page 11 (where the local g is plotted as a function of the local r) is modified.
b) How does this assumption modify the 84 minute period in all the applications discussed?

IX. Inertial Navigation

An inertial navigation or inertial guidance device is a combination of three essential elements: first a platform which keeps its direction fixed in inertial space (by means of gyroscopes now or possibly by other means in the future); second a number of accurate accelerometers, in various directions; third a computer which integrates the output of the accelerometers and thus keeps track of velocity, location and angular position in space of the accelerometers. The details of the arrangement of these elements into the whole differ with the application. For ship navigation on or near the surface of the earth (the ocean) there are two accelerometers only, one always directed NS, the other always EW. When the ship moves from place to place, or even when it is standing still at the dock these NS and EW directions continually change with respect to absolute space, caused both by the ship's motion and by the rotation of the earth. One of the duties of the computer is to keep track of all this and to give commands to servomotors which continually change the position of the accelerometer platform relative to the steady inertial platform. A third accelerometer (in the vertical direction) serves no purpose, since it would measure g only. Hence only two are used, while for space navigation three are necessary. The accuracy whereby the accelerations have to be measured is phenomenal, a very slight error in the measured acceleration becomes a very large error in position when integrated twice over long time intervals.

Problem 12

If one (horizontal) accelerometer has a fixed error, reading the

acceleration high by one millionth of g, calculate the error in the ship's position after 3 days.

One very serious source of possible error is a slight tilt of the accelerometer platform from the horizontal. In that case the accelerometer not only feels the acceleration of the ship, but also a component of g, which it cannot distinguish from a ship's acceleration. When adjusting the apparatus at dockside prior to a voyage the accelerometers have to be set horizontal to give zero reading. However this can be done to a certain accuracy only and the ship may go out with an initial tilt of the accelerometer of one part in a million or so. On a flat earth this error would remain right in the system with the disastrous results of the above Problem 12. However on a curved earth the tilt at dockside does not remain constant. This is because the accelerometer measures an acceleration (due to tilt) which it interprets as ship's acceleration and hence as ship's displacement after two integrations. When the computer believes that the ship is a few miles from the dock to which it is actually moored it concludes that the platform has to be turned through a small angle to be perpendicular to the new vertical in the new (spurious) location. The result is that with a live computer circuit the table oscillates about the true horizontal position (with the inevitable 84 minute period). The computer concludes that the ship sails back and forth with an 84 minute period, instead of concluding that it sails in one direction with a constant (small) steady acceleration, piling up a fantastic velocity and displacement in the course of days.

The analysis is as follows: with the ship at dockside, while the true actual acceleration is zero, let the tilt angle be φ (a function of time) with the consequent indicated acceleration $g\varphi$. The computer integrates this into a displacement we will call x (also a function of time). The computer then gives a command to tilt the platform back through angle x/R (being the angle between the true verticals at the dock and at the presumed position x). If the original tilt at time zero is φ_o then the actual tilt angle of the platform is $\varphi = \varphi_o - x/R$, and the indicated acceleration is

$$\ddot{x} = g\varphi = g(\varphi_o - x/R)$$

or

$$\ddot{x} + g/R\ x = g\varphi_o \qquad (1)$$

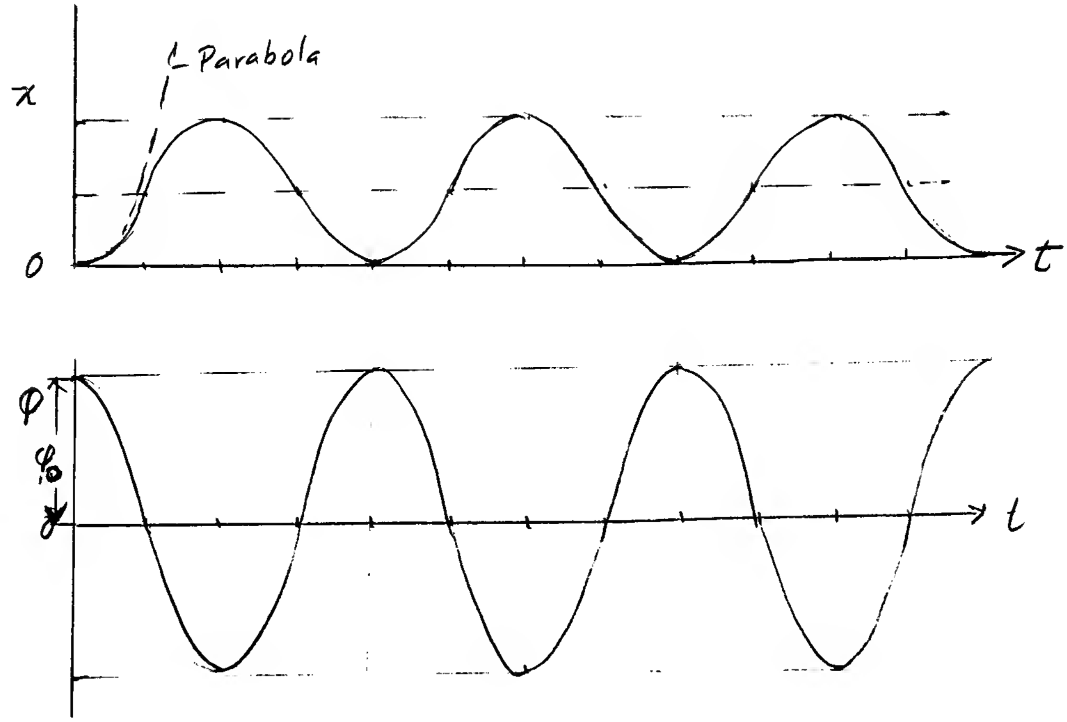
Thus the indicated position is seen to oscillate with period $\omega^2 = g/R$. To find the differential equation of the tilt angle φ we differentiate (1) twice

$$\ddddot{x} + \frac{g}{R}\ddot{x} = 0$$

with $\ddot{x} = g\varphi$ this becomes

$$\ddot{\varphi} + \frac{g}{R}\varphi = 0 \tag{2}$$

The initial conditions are that at $t = 0$ the platform angle $\varphi = \varphi_o$, and the indicated distance from dockside $x = 0$, and of course $\dot{\varphi} = \dot{x} = 0$. With this the graphs of $x(t)$ and $\varphi(t)$ are as shown below.



In case the earth were flat no Schuler tuning effect would exist and the $x(t)$ diagram would be the osculating parabola to the sine curve, shown dotted in the graph. It is seen that the automatic Schuler tuning saves the day and that without it inertial navigation on earth over prolonged periods of time would be next to nigh impossible.

Problem 13

Calculate the maximum indicated displacement for an initial error angle of:

a) One part in a million (10^{-6} radians).
b) One part in a billion.

Schuler Tuning of the Gyroscopic Ship's Compass.

We start with a reprint of the author's textbook "Mechanics", available at Dover Publications, New York. The reprint describes the compass construction in physical terms without mathematics:

62. The Gyroscopic Ship's Compass. The gyroscopic ship's compass is one of the most ingenious mechanical devices known and is a fitting subject for the last article of this book. Its operation is almost miraculous. It feels the rotation of the earth, one revolution in 24 hours, and it feels this rotation correctly, even when mounted in a ship that pitches and rolls violently in a storm. When someone tampers with the compass and makes it point in a wrong direction, the compass distinguishes between the earth's rotation and the ship's motions, corrects its own error, and after a few hours points to the true north again. Obviously a mechanism that can do all of this cannot be explained in a few words, and in order to understand its operation we consider some preliminary stages before coming to the actual construction of Fig. 300.

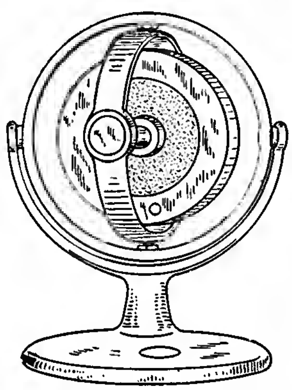


FIG. 292. A gyroscopic disk mounted in three well-balanced gimbal rings preserves its direction in space and therefore is a primitive form of compass.

First we imagine a gyroscopic disk mounted in three gimbal rings (Fig. 292). If the base of the apparatus is held fixed, the gyro axis can be made to point in any direction in space. If the three gimbal axes all intersect in one point and that point is the center of gravity of the gyro rotor as well as of each gimbal ring individually, and if the gyro rotor axis is a principal axis of inertia, then no moments act on the gyro rotor. If it is set spinning fast, it will therefore preserve its direction in space, independent of the motion of the base. In a sense this device could be used as a compass: if the gyro axis is pointed toward Polaris, it will stay there. However, there are three things wrong with this kind of compass:

a. It points north, but it is not horizontal, except at the equator.

b. Friction in the gimbal bearings causes small torques, which will gradually throw it off.

c. It is not self-correcting.

If the compass of Fig. 292 were mounted at the equator and pointed north, *i.e.*, parallel to the axis of rotation of the earth, it would stay in that position indefinitely, except for bearing friction. Now let us

consider Fig. 293, which represents the earth as seen by an observer in space far above the North Pole. The earth rotates from west to east once every 24 hours. If the compass of Fig. 292 is mounted in position 1 at the equator, pointing east and horizontal, it will find itself 6 hours later in position 2, pointing vertically up; after another 6 hours, in position 3 pointing west, and finally in position 4, pointing down. After this is understood, we imagine our compass in Fig. 293 started at position 1, not pointing east as shown, but practically north with a small easterly error. Then at 2 we have a small upward error, at 3 a small westerly error, and at 4 a small downward error.

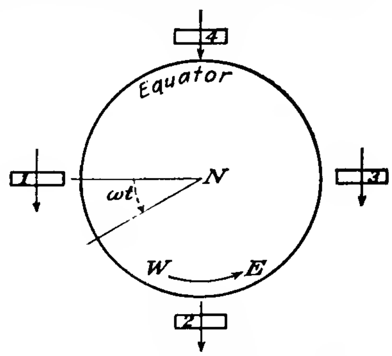


FIG. 293. The earth seen from far above the North Pole.

Next we proceed to the construction of Fig. 294, which is closer to the actual compass than Fig. 292. The gyroscopic disk is still mounted in the same three gimbal rings, but now a pendulous mass has been added. When the gyro axis is horizontal this pendulous mass has no influence, but as soon as the gyro axis has an upward deviation, the pendulum exerts a torque, which drives the end point of the $\mathfrak{M}$ vector to the west. Similarly when the $\mathfrak{M}$ vector points north and is dipped down slightly, the pendulum torque drives the top of the $\mathfrak{M}$ vector toward the east.

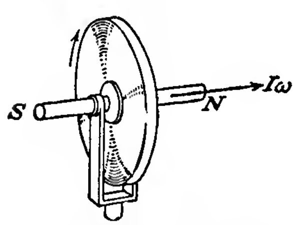


FIG. 294 A gyroscopic disk carrying a gravity pendulum is the second step in the explanation of the marine compass.

Now imagine that a sheet of paper is held perpendicular to the $\mathfrak{M}$ or gyro axis at a distance of several feet north of the compass, and that we mark on this paper the point where the gyro axis intersects it. The result is Fig. 295, showing a coordinate system (east-west; up and down) with an origin O, and the coordinate axes marked in degrees. If the compass (Fig. 294), still at the equator, is started with an easterly deviation of 4 deg from the true-north horizontal position, the gyro axis intersects the paper at point A. The rotation of the earth, by Fig. 293, introduces an upward deviation, so that the axis moves to point B of Fig. 295. Then the pendulum torque takes hold and drives it to the west. We thus reach point C, where the tangent to the curve

is horizontal, because there is no longer an easterly deviation at that point. The pendulum torque continues its westward push, so that soon there is a westerly deviation, which by the earth's rotation causes a downward motion. Thus we arrive at D, then at E and back to A.

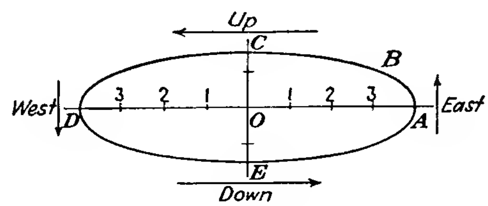


Fig. 295. Motion of the end point of the angular momentum vector of the compass Fig. 294.

The period of this elliptic motion depends on the intensity of the pendulum torque, and in an actual compass it is so adjusted that one full cycle takes 84 minutes, which is the period $T = 2\pi \sqrt{R/g}$ of a simple pendulum with a length equal to the radius of the earth. The practical reason for this particular relation is too complicated for explanation in this text.

Thus we see that the compass of Fig. 294, when placed at the equator with a deviation, will move around the desired true-north horizontal position, but will never reach it: it is not yet a satisfactory compass. In order to make it satisfactory, the diagram of Fig. 295 has to be modified to Fig. 296, by the introduction of a "damping torque" into the compass. The difference between the two figures can best be seen at point C. In the first figure the curve there is parallel to the east-west axis; in the second figure it is not. In the first figure a deviation causes a reaction pushing the compass in a direction 90 deg from the deviation. In the second figure this is still so, but in addition there is a reaction diminishing the original deviation directly. At point C in Fig. 296 the deviation is upward only, but the curve at C has a westward as well as a downward slope. In this manner the ellipse of Fig. 295 is transformed into the spiral of Fig. 296. The latter diagram describes a satisfactory compass, because a deviation gradually corrects itself.

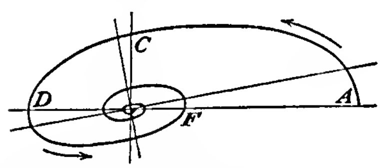


Fig. 296. Modification of the motion diagram Fig. 295 by the introduction of damping.

In order to introduce the damping torque into the mechanism, Fig. 294 has to be modified radically into Fig. 297, in which the gyro rotor

A is mounted in a casing B, which is held in a gimbal ring C by means of two hinges. The ring C is suspended from a piece D by means of a flexible wire or very thin rod. This part D is known as the *phantom element.* It in turn carries the pendulum E, which thus is not carried by the gyro rotor directly. The pendulum E is coupled to the rotor casing B by means of a pin F, which is drawn in eccentrically, but which for the moment may be considered to be just in the vertical center line.

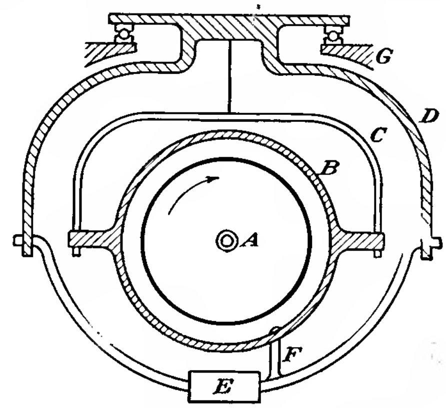


FIG. 297. Third step in the explanation of the gyroscopic marine compass.

The phantom element rests on a piece G through a ball bearing. The piece G is carried through a complete set of gimbal rings by the ship.

The connection between E and B through the pin F is such that the pin is rigidly built into E but rides in a long slot of B, such that the force between F and B can only be perpendicular to the paper. If for any reason the assembly ABC tends to swing to the right in the plane of the paper, this can take place without any restraint on account of the slot. If on the other hand ABC tends to swing out of and into the paper with respect to EF, the pin couples the two motions.

Then there is another thing yet. If the assembly ABC turns (about a vertical axis) with respect to D, twisting the supporting wire, an electric motor is started which turns the phantom D on its ball bearings on G so as to set it in line with ABC again. Thus the phantom always follows the rotations of ABC about the vertical axis and no twist in the wire can be permanent. This turning is regulated by means of a pair of electric contacts and is so sensitive that the following action is started as soon as the twist between C and D is $\frac{1}{4}$ deg.

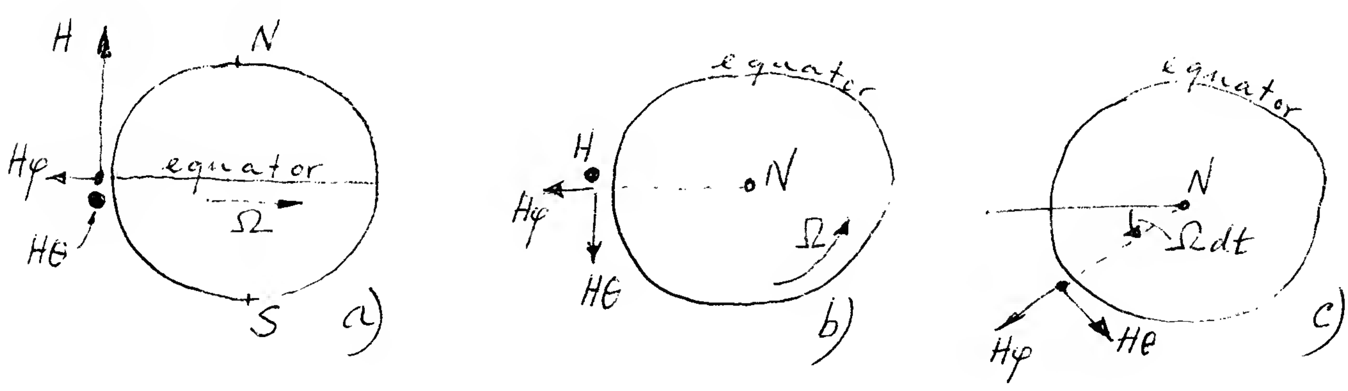
It is easy to see that the compass with the *central* pin F acts exactly like Fig. 294. When it is deviated east, the earth tips the point of the $\mathfrak{M}$ vector of A upward. Then the casing B pushes against the

pendulum pin F, and a torque is created on A about a horizontal axis passing through the two bearings of B. This torque has exactly the same sense as the gravity torque of Fig. 294. The apparatus, although much more complicated, accomplishes exactly the same thing as Fig. 294.

But it is now capable of introducing a *damping* as specified in Fig. 296, simply by setting the pin F somewhat to the right shown in Fig. 297. When that is done, the force between F and B (which is perpendicular to the paper) not only gives a moment about the horizontal axis but also about the *vertical* center line AE. In the case of Fig. 297 the main $\mathfrak{M}$ vector points into the paper away from the reader. When the point of that vector is deflected upward, the casing B pushes the pin F into the paper and consequently F pushes B out of the paper. This force on B gives a couple about the vertical center line represented by a vector pointing downward. Thus an *upward* deviation of the compass causes a *downward* precession as desired.

So far the reprint. It continues to describe some further technical details that are not essential to the understanding of our subject. Hence we now proceed to a mathematical analysis of the operation of the compass Fig. 297, as interpreted by the diagrams Figs. 295 or 296. We consider the compass on a ship at the equator of the Earth. The analysis at some other location of latitude $\lambda \neq 0$ is more complicated but not essentially different.

We call the (small) upward error angle φ and the (small) Eastward error angle Θ, plotted vertically and horizontally in Fig. 295/296. The earth turns from West to East with angular speed $\Omega = 2\pi$ radians/24 hours.



The sketch above shows the angular momentum vector H (designated as $\mathcal{H}$ in the book reprint) in its three components (a large one H northward, two small ones $H\varphi$ upward and $H\theta$ eastward) in two projections a) and b). The sketch c) shows the H vector a short time dt later. The rates of change of H are seen to be:

in the upward direction $\quad H\dot{\varphi} - H\theta\Omega$
in the eastward direction $\quad H\dot{\theta} + H\varphi\Omega$

in which the first terms are caused by the change in length of the vectors and the second terms by the change in direction of the vectors due to the earth rotation rate Ω. The moment acting on the gyroscope by gravity is $wl\varphi$ westward and $wa\varphi$ downward if w is the weight of E in Figure 297, l is the vertical distance between A and E and a is the horizontal offset distance between the pin F and a vertical center line AE in Figure 297.

If now the compass is accelerated Northward with an acceleration $\ddot{x}$ a southward inertia force $(w/g)\ddot{x}$ acts on the pendulum mass and two additional moments act on the gyrowheel: $(w/g)\ddot{x}l$ westward and $(w/g)\ddot{x}a$ downward. The equations of motion thus are:

$$\text{in the up-direction} \qquad H\dot{\varphi} - H\theta\Omega = -wa\varphi - \frac{w}{g}\ddot{x}a \qquad (1)$$

$$\text{in the east-direction} \qquad H\dot{\theta} + H\varphi\Omega = -wl\varphi - \frac{w}{g}\ddot{x}l \qquad (2)$$

Eliminate θ by solving for $\dot{\theta}$ from the second of these and substituting into the differentiated first one:

$$\ddot{\varphi} + \frac{wa}{H}\dot{\varphi} + \left(\Omega^2 + \frac{wl\Omega}{H}\right)\varphi = -\frac{wl\Omega}{gH}\ddot{x} - \frac{wa}{gH}\dddot{x}$$

A similar equation can be obtained for θ instead of φ. We see that φ describes a damped vibration, forced by the acceleration $\ddot{x}$ and $\dddot{x}$. The natural frequency ω_n of the free motion is:

$$\omega_n^2 = \Omega^2 + \frac{wl\Omega}{H}$$

Jumping ahead on the conclusion to be reached soon, that the natural period is 84 minutes ($= 2\pi/\omega_n$), and noting that Ω, the earth's rotation, has a period of 24 hours, which is 17.2 times as slow, we recognize that the Ω^2 term is $(17.2)^2 = 295$ times as small as ω_n^2. Hence, accepting an error of one part in 295 we write

$$\omega_n^2 = \frac{wl\Omega}{H} \qquad (3)$$

for the natural frequency of the compass motion.

Considering only the case of constant acceleration $\ddot{x}$ (so that $\dddot{x} = 0$) the equation of the up and down φ motion is:

$$\ddot{\varphi} + \frac{wa}{H}\dot{\varphi} + \frac{wl\Omega}{H}\varphi = -\frac{wl\Omega}{gH}\ddot{x} \tag{4}$$

When starting the acceleration $\ddot{x}$ from the position of rest, i.e., from $\varphi = \dot{\varphi} = 0$ we find

$$\ddot{\varphi} = -\frac{wl\Omega}{gH}\ddot{x}$$

for the acceleration of the hang back angle φ. The minus sign indicates that a Northward acceleration, causing a Southward hang back will tip the needle downwards, which is a negative φ. The $\ddot{x}$ acceleration causes a displacement x with a change ψ in the direction of the local vertical, where $\psi = -x/R$ or $\ddot{\psi} = -\ddot{x}/R$.

Schuler compensation or insensitivity to acceleration occurs when $\varphi = \psi$ or $\ddot{\varphi} = \ddot{\psi}$ or

$$-\frac{wl\Omega}{gH}\ddot{x} = -\frac{\ddot{x}}{R}$$

$$\therefore \frac{wl\Omega}{gH} = \frac{g}{R}, \quad \text{and by Eq. (3)} \quad \omega_n^2 = \frac{g}{R},$$

which is our old friend, q. e. d.

Problem 14

From Eq. (1), considering Eq. (3) show that for no acceleration ($\ddot{x} = 0$) and for no damping ($a \doteq 0$) the diagram Figure 295 has the shape $OA/OC = \theta_{max}/\varphi_{max} = \omega_n/\Omega = 17.2$.

Problem 15

Assuming $\ddot{x} = 0$, Equation 4 determines the Figure 296. If we want one half critical damping ($c/c_c = 0.5$) show that the ratio $a/\ell = \Omega/\omega_n = 1/17.2$. Thus for a practical compass the sidewise offset of the pin F in Figure 297 is very small.

Answers to Problems

2. a) 1.7×10^{-6} inches.

   b) $f = 8.3 \times 10^{-5}\varphi_o = 8.3 \times 10^{-5}\ \ddot{x}_o/g$.

   Hence for a friction coefficient of 8.3 per cent (a practical value) a Schuler tuned pendulum can be subjected to $\ddot{x}_o = 1000g$ without breaking the friction lock of the hinge. This illustrates the statement of page 3 that the pendulum is "physically possible, but not practically feasible."

4. $f = 30\ \ddot{x}_o/g$, large in comparison to the answer to Problem 2b. Hence the Schuler tuned gyropendulum is a more satisfactory vertical indicating device than the Schuler tuned plain physical pendulum.

7. a) The particle weight divided by 4000.

   b) $f = 1/4000$.

8. The curve always is symmetrical about the horizontal and vertical axes. Hence only a quarter has be to calculated:

$\pm\ x/x_o$	= 0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
$\pm\ y$ (for $x_o \ll R$)	=1.000	.995	.980	.954	.917	.866	.800	.714	.600	.436	0
$\pm\ y$ (for $x_o = 2R$)	=1.050	1.032	.980	.906	.815	.720	.620	.518	.406	.278	0

10. 61 radii

11. a) The g-value rises linearly from zero at the center of the core to $g_o = 32.2$ at the edge of the core, then remains constant at 32.2 throughout the mantle. The value 0.618 R was specially chosen to produce this (special) simple result.

    b) For motions outside the earth (satellites and tangent surface track) the 84 minute period is preserved unchanged. For motions inside the earth the differential equations become non-linear and the 84 minute period is modified. In particular for case IV, the diametral hole along the axis of rotation, the 'spring' is no longer linear, but follows the diagram of Problem 11a. The non-linear problem can be solved by the usual methods (for example Den Hartog's Mechanical Vibrations, 4th Edition, page 360/361) with a resulting period of 78 minutes.

12. 206 Miles
    The submarines navigating under the polar ice cap had their position indicated to within a few miles after a 3-day voyage.

13. a) 42 feet

    b) 1/2 inch

MATHEMATICS

To Everywhere in 42 Minutes

Although he is now 36, and a mathematician for Sylvania, Paul Cooper has never lost his boyhood enthusiasm for the faneiful science-fietion stories of Jules Verne. While musing about *Journey to the Center of the Earth* several months ago, Cooper himself took off on a mathematical flight of faney that more than rivals Verne's most imaginative work. By crisscrossing the earth with subterranean tunnels, the freewheeling mathematician proposes in the current issue of the *American Journal of Physics*, man could aehieve intereontinental travel at ballistic missile speed.

Verne's underground hike would have taken far less time, says Cooper, if he had simply fallen into a frictionless tunnel bored through the earth's center. Accelerated by the force of gravity on the first half of his trip, he would have gained just enough kinetie energy to coast up to the other side—against the pull of gravity—in a total time of only 42.2 minutes.

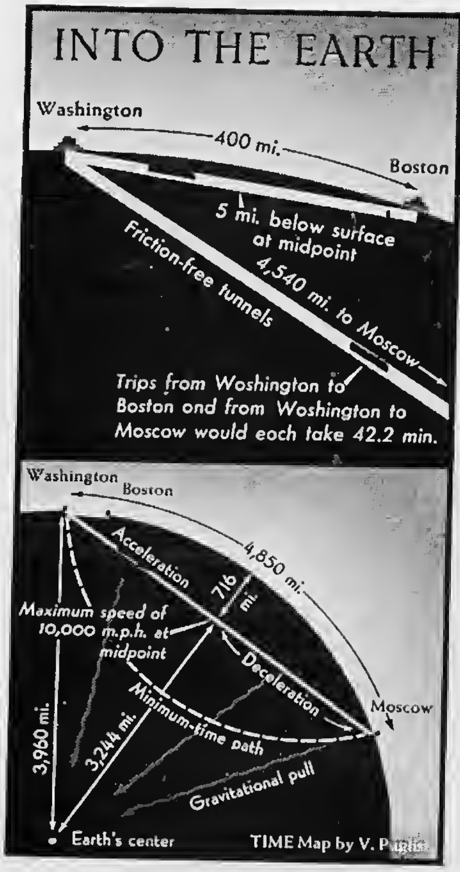
Universal Timetable. Fascinated by his initial calculation, Cooper worked out a formula for the time required for an object to fall through a straight-line tunnel bored between any two points on the surface of the earth. Surprisingly, no matter how elose or far apart the two points were, the time turned out to be constant: 42.2 minutes.

According to Cooper's equations, by "dropping" in airless, frictionless, straight-line tunnels, passenger vehicles powered only by the pull of gravity could theoretically travel between Washington and Moscow, which are 4,850 surface miles apart, in the same time it would take them to travel from Washington to Boston, only 400 miles away. "One ean envisage a transportation system without timetables," says Cooper, tongue in cheek, "with the world's cities linked by tunnels, the departure time universally on the hour, and the arrival time 42.2 minutes later."

Gravity-Powered Travel. To be sure, some formidable obstacles would have to be overcome before his seheme eould become reality. At its midpoint, a Washington-Boston tunnel would be five miles below the surface of the earth—a teehnieally difficult and prohibitively costly bit of construction. In addition, the subterranean temperature at a five-mile depth might be as high as 265° F., and a passenger vehicle would need an immense eooling system. Finally, because a perfect vaeuum could not be created within the tunnel, and because the vehiele would probably have to ride on some sort of rail, friction would slow it down—leaving it with insuffieient kinetic energy to complete its trip without a source of additional power. In a long-distanee Washington-Moscow tunnel, which at its midpoint would dip some 716 miles below the earth's surface, the problems would surely be magnified beyond solution.

Undaunted by such practicalities, Cooper has also set up and solved by computer a set of differential equations for eurved tunnels that would provide minimum gravity-powered travel time between any two cities on earth. These tunnels would swoop into the ground at steeper angles and penetrate to even greater depths. Though travel times would vary, all would be less than the 42.2 minutes required for straight-line trips.

Cooper has let his imagination soar even farther. Using different radii and gravitational forces in his formulas, he has laid out the mathematical groundwork for extraterrestrial travel networks. According to his calculations, straight-line tunnel travel between any two surfaee locations would be 53 minutes on the moon, 49 on Mars.



TIME, FEBRUARY 11, 1966